Large deviation expansions for the coefficients of random walks on the general linear group

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(Ann. Probab. 51(4): 1380-1420, 2023)

The 18-th Workshop on Markov Processes and Related Topics

July 31, 2023

Outline

1. Background
2. Large deviation expansions for the coefficients
3. Key ideas of the proof
4. Applications to multivariate perpetuity sequences

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Random walks on the general linear group

- For any integer $d \geq 1$, set $V = \mathbb{R}^d$.
  (V can also be $\mathbb{C}^d$ or $\mathbb{K}^d$, where $\mathbb{K}$ is a local field.)
  Fix an orthonormal basis $e_1, \ldots, e_d$ of V.
  Let V^* be the dual vector space of V.
- Let $(g_n)_{n \geq 1}$ be a sequence of independent and identically distributed random elements with law μ on the general linear group $\mathrm{GL}(V)$.

  Consider the random walk (or products of random matrices)

$$G_n = g_n \ldots g_1.$$

- In this talk, we are interested in large deviations for the coefficients $\langle f, G_n v \rangle$, where $v \in V \setminus \{0\}$ and $f \in V^* \setminus \{0\}$.
  In particular, taking $v = e_j$ and $f = e_i^*$, we get the (i,j)-th coefficient $G_n^{i,j} := \langle e_i^*, G_n e_j \rangle$.

Moment condition and condition (IP)

For $g \in \mathrm{GL}(V)$, set $\|g\| = \sup_{v \in \mathbb{R}^d \setminus \{0\}} \frac{\|gv\|}{\|v\|}$ and $N(g) = \max\{\|g\|, \|g^{-1}\|\}$.

Exponential moment condition

There exists a constant $\eta > 0$ such that $\mathbb{E}[N(g)^\eta] < \infty$.

We say μ has p-th moment if $\mathbb{E}[\log^p N(g)] < \infty$ for $p > 0$.

- Let Γ_μ be the smallest closed semigroup generated by $\operatorname{supp} \mu$.
- A matrix $g \in \mathrm{GL}(V)$ is called proximal if it has an eigenvalue λ with multiplicity one and all other eigenvalues of g have modulus strcitly less than $|\lambda|$.

Condition (IP)

(i)(Strong irreducibility) No finite union of proper subspaces of $\mathbb{R}^d$ is Γ_μ-invariant.
(ii)(Proximality) Γ_μ contains a proximal matrix.

Laws of large numbers: (1)

Theorem (Guivarc'h-Raugi, PTRF 1985; Benoist-Quint, Springer 2016)

Assume the exponential moment condition and condition (IP). Then, for any $v \in V \setminus \{0\}$ and $f \in V^* \setminus \{0\}$,

$$\lim_{n \to \infty} \frac{1}{n} \log |\langle f, G_n v \rangle| = \lambda \quad \text{a.s.} \qquad \boxed{\text{SLLN}} \tag{1.1}$$

where λ is a constant called the first Lyapunov exponent of μ.

Note: Furstenberg-Kesten (Ann. Math. Statist. 1960) established the SLLN for $\|G_n\|$: if $\mathbb{E} \log^+ \|g_1\| < +\infty$, then

$$\lim_{n \to \infty} \frac{1}{n} \log \|G_n\| = \lambda \quad \text{a.s.}$$

It can be seen as a corollary of Kingman's subadditive ergodic theorem (Ann. Probab. 1973).

Laws of large numbers: (2)

Assume condition (IP).

Theorem (Grama-Liu-X., arXiv 2021)

If $\int_{\mathrm{GL}(V)} \log N(g)\mu(dg) < \infty$, then, as $n \to \infty$, uniformly in $v \in V$ and $f \in V^*$ with $\|v\| = \|f\| = 1$,

$$\frac{1}{n} \log |\langle f, G_n v\rangle| \to \lambda \quad \text{in probability and in } L^1. \qquad \boxed{\text{WLLN}} \qquad (1.2)$$

Moreover, if $\int_{\mathrm{GL}(V)} \log^2 N(g)\, \mu(dg) < \infty$, then the SLLN for $\langle f, G_n v\rangle$ holds.

Open question: whether the SLLN holds true for $\langle f, G_n v\rangle$ under the first moment condition.

Benoist-Quint (Random walks on groups, 2016): "It is plausible."

Benoist-Quint: Stationary measures and invariant subsets of homogeneous spaces (I, II, III), Ann. Math. 2011, JAMS 2013, Ann. Math. 2013.

Central limit theorem

Theorem (Guivarc'h-Raugi, PTRF 1985)

Assume the exponential moment condition and condition (IP). Then, for any $t \in \mathbb{R}$, $v \in V \setminus \{0\}$ and $f \in V^* \setminus \{0\}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log |\langle f, G_n v\rangle| - n\lambda}{\sigma\sqrt{n}} \leq t \right) = \Phi(t), \tag{1.3}$$

where $\sigma^2 > 0$ is the asymptotic variance and $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{u^2}{2}} du$.

Benoist-Quint (AOP 2016) proved (1.3) under the second moment condition and condition (IP). The proof borrows some ideas from

Bourgain, J., Furman, A., Lindenstrauss, E., Mozes, S.: Stationary measures and equidistribution for orbits of nonabelian semigroups on the torus. J. Am. Math. Soc. 24, 231-280 (2011)

Related results

Assume condition (IP).

1. Berry-Esseen bounds:
   - Under the exponential moment condition, Cuny-Dedecker-Merlevède-Peligrad (Comptes Rendus. Mathématique 2022) obtained the rate $\frac{c \log n}{\sqrt{n}}$.
   - Under the exponential moment condition, Dinh-Kaufmann-Wu (J. Inst. Math. Jussieu 2022) improved it to be $\frac{c}{\sqrt{n}}$.
   - Under the third moment condition, Dinh-Kaufmann-Wu (PTRF 2023) obtained the rate $\frac{c}{\sqrt{n}}$ for 2×2 matrices.
2. First-order Edgeworth expansion: by Grama-Liu-X. (arXiv 2021).
3. Local limit theorem: by Grama-Quint-X. (AIHP 2022).
4. Moderate deviations: by Grama-Liu-X. (SPA 2023).

Open problem: how to obtain the Berry-Esseen bound and the first-order Edgeworth expansion under the third moment condition.

Objectives and known result

We study the rate of convergence in the law of large numbers.

1. Bahadur-Rao type large deviations: for $q > \lambda$, as $n \to \infty$,

$$\mathbb{P}(\log |\langle f, G_n v\rangle| \geq nq) \sim ?$$

2. Petrov type large deviations: with a "small" perturbation $|l| \leq l_n$ on q, where $l_n \to 0$ as $n \to \infty$,

$$\mathbb{P}\big(\log |\langle f, G_n v\rangle| \geq n(q + l)\big) \sim ?$$

Theorem (Benoist-Quint, 2016)

For any $v \in V \setminus \{0\}$ and $f \in V^* \setminus \{0\}$, there exist $c, C > 0$ such that

$$\mathbb{P}\big(\log |\langle f, G_n v\rangle| \geq nq\big) \leq Ce^{-cn}.$$

Outline

Banach space and Markov chain

1. Let $\mathscr{C}(\mathbb{P}(V))$ be the space of continuous complex-valued functions on $\mathbb{P}(V)$. For $\gamma > 0$ and $\varphi \in \mathscr{C}(\mathbb{P}(V))$, set
$$\|\varphi\|_\gamma := \|\varphi\|_\infty + [\varphi]_\gamma, \ \|\varphi\|_\infty := \sup_{x \in \mathbb{P}(V)} |\varphi(x)|, \ [\varphi]_\gamma := \sup_{x,x' \in \mathbb{P}(V)} \frac{|\varphi(x) - \varphi(x')|}{d(x,x')^\gamma},$$
where $d(x,x') = \frac{\|v \wedge v'\|}{\|v\|\|v'\|}$ for $x = \mathbb{R}v$ and $x' = \mathbb{R}v'$.
Introduce the Banach space $\mathscr{B}_\gamma := \{\varphi \in \mathscr{C}(\mathbb{P}(V)) : \|\varphi\|_\gamma < +\infty\}$.
2. Denote $X_0^x := x$ and $g \cdot x := \mathbb{R}gv$ for $g \in \mathrm{GL}(V)$ and $x = \mathbb{R}v \in \mathbb{P}(V)$.
Let $X_n^x := G_n \cdot x = \mathbb{R}G_n v$ for $x = \mathbb{R}v \in \mathbb{P}(V)$ with $v \in V \setminus \{0\}$.
Then $(X_n^x)_{n \geq 0}$ is a Markov chain on the projective space $\mathbb{P}(V)$.
3. The transfer operator of the Markov chain $(X_n^x)_{n \geq 0}$ is given by
$$P\varphi(x) = \int_{\mathrm{GL}(V)} \varphi(g \cdot x) \mu(dg).$$
4. Spectral gap (Le Page 1982): there exists a unique invariant probability measure ν on $\mathbb{P}(V)$ satisfying
$$\|P^n \varphi(x) - \nu(\varphi)\|_\gamma \leq C e^{-cn}.$$

Laplace transform

1. Let $I_\mu = \{s \geq 0 : \mathbb{E}(\|g_1\|^s) < \infty\}$ and let I_μ° be its interior.
2. Laplace transform of the Markov random walk: for $s \in I_\mu$, define

$$P_s\varphi(x) = \int_{\mathrm{GL}(V)} e^{s\sigma(g,x)}\varphi(g \cdot x)\mu(dg), \quad x \in \mathbb{P}(V),$$

where, for $g \in \mathrm{GL}(V)$ and $x = \mathbb{R}v \in \mathbb{P}(V)$,

$$\sigma(g,x) = \log \frac{\|gv\|}{\|v\|}$$

is a cocycle.

Guivarc'h-Le Page's spectral gap theory (2016)

1. For $s \in I_\mu$, set

$$\kappa(s) = \lim_{n\to\infty} \left(\mathbb{E}\|G_n\|^s\right)^{\frac{1}{n}}.$$

Define $\Lambda(s) = \log\kappa(s)$ and its Legendre transform:

$$\Lambda^*(q) = \sup_{s\in I_\mu}\{sq - \Lambda(s)\}, \quad q \in \Lambda'(I_\mu).$$

2. There exist a unique eigenfunction r_s and a unique eigenmeasure ν_s:

$$P_s r_s(x) = \kappa(s) r_s(x), \qquad \nu_s P_s(\varphi) = \kappa(s)\nu_s(\varphi).$$

Moreover,

$$r_s(x) = \int_{\mathbb{P}(V^*)} \delta(y,x)^s \nu_s^*(dy),$$

where $\delta(y,x) = \frac{|\langle f,v\rangle|}{\|f\|\|v\|}$ for $x = \mathbb{R}v \in \mathbb{P}(V)$ and $y = \mathbb{R}f \in \mathbb{P}(V^*)$.

Bahadur-Rao type precise LD for the coefficients

- Assume (IP) and exponential moments.
- Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$.
- Notation: $\sigma_s = \Lambda''(s)$, $x = \mathbb{R}v \in \mathbb{P}(V)$ and $y = \mathbb{R}f \in \mathbb{P}(V^*)$.

Theorem (Grama-Liu-X., AOP 2023, Bahadur-Rao type LD)

Uniformly in $v \in V$ and $f \in V^*$ with $\|v\| = \|f\| = 1$,

$$\mathbb{P}\Big(\log |\langle f, G_n v\rangle| \geq nq \Big) \sim \frac{r_s(x) r_s^*(y)}{\nu_s(r_s)} \frac{\exp\left(-n\Lambda^*(q)\right)}{s\sigma_s\sqrt{2\pi n}}.$$

Here, r_s^* is the eigenfunction of the dual operator P_s^* defined by

$$P_s^*\varphi(y) = \int_{\mathrm{GL}(V)} e^{s\sigma(g^*,y)} \varphi(g^* \cdot y) \mu(dg), \quad y \in \mathbb{P}(V^*),$$

where g^* denotes the adjoint automorphism of $g \in \mathrm{GL}(V)$.
This improves the result of Benoist-Quint (2016):

$$\mathbb{P}\Big(\log |\langle f, G_n v\rangle| \geq nq \Big) \leq Ce^{-cn}.$$

Petrov type precise LD for the coefficients

- Assume (IP) and exponential moments.
- Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$; $x = \mathbb{R}v \in \mathbb{P}(V)$ and $y = \mathbb{R}f \in \mathbb{P}(V^*)$.
- Let $\{l_n\}$ be any positive sequence such that $l_n \to 0$ as $n \to \infty$.

Theorem (Grama-Liu-X., AOP 2023, Petrov type LD)

Uniformly in $|l| \leq l_n$, $v \in V$ and $f \in V^*$ with $\|v\| = \|f\| = 1$,

$$\mathbb{P}\Big(\log |\langle f, G_n v \rangle| \geq n(q + l) \Big) \sim \frac{r_s(x) r_s^*(y)}{\nu_s(r_s)} \frac{\exp\left(-n\Lambda^*(q+l)\right)}{s\sigma_s\sqrt{2\pi n}}.$$

We have $\Lambda^*(q+l) = \Lambda^*(q) + sl + \frac{l^2}{2\sigma_s^2} - \frac{l^3}{\sigma_s^3}\zeta_s\left(\frac{l}{\sigma_s}\right)$, where ζ_s is the Cramér series given by

$$\zeta_s(t) = \frac{\gamma_{s,3}}{6\gamma_{s,2}^{3/2}} + \frac{\gamma_{s,4}\gamma_{s,2} - 3\gamma_{s,3}^2}{24\gamma_{s,2}^3} t + \frac{\gamma_{s,5}\gamma_{s,2}^2 - 10\gamma_{s,4}\gamma_{s,3}\gamma_{s,2} + 15\gamma_{s,3}^3}{120\gamma_{s,2}^{9/2}} t^2 + \ldots$$

with $\gamma_{s,k} = \Lambda^{(k)}(s)$ for any $k \geq 1$.

Bahadur-Rao-Petrov type LD: lower tails

- Assume (IP) and exponential moments.
- Let $\{l_n\}$ be any positive sequence such that $l_n \to 0$ as $n \to \infty$.

Theorem (Grama-Liu-X., AOP 2023, Petrov type LD)

There exists a constant $s_0 > 0$ such that for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, uniformly in $|l| \leq l_n$, $v \in V$ and $f \in V^*$ with $\|v\| = \|f\| = 1$,

$$\mathbb{P}(\log |\langle f, G_n v\rangle| \leq n(q + l)) \sim \frac{r_s(x) r_s^*(y)}{\nu_s(r_s)} \frac{\exp\left(-n\Lambda^*(q+l)\right)}{-s\sigma_s\sqrt{2\pi n}}.$$

- Taking $l = 0$, we get the Bahadur-Rao type LD result.
- Open problem: as $n \to \infty$,

$$\mathbb{P}(\log |\langle f, G_n v\rangle| \leq nq) \sim ?$$

in the case when $q < \lambda$ is arbitrary (q is not necessarily sufficient close to the Lyapunov exponent λ).

Applications to local limit theorems with LD

- Assume (IP) and exponential moments.
- Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$; $x = \mathbb{R}v \in \mathbb{P}(V)$ and $y = \mathbb{R}f \in \mathbb{P}(V^*)$.
- Let $\{l_n\}$ be any positive sequence such that $l_n \to 0$ as $n \to \infty$.

Theorem (Grama-Liu-X., AOP 2023, Local limit theorem with large deviations)

There exists a sequence $\Delta_n > 0$ converging to 0 as $n \to \infty$ such that, uniformly in $|a| \leq nl_n$, $\Delta \in [\Delta_n, nl_n]$, $v \in V$ and $f \in V^*$ with $\|v\| = \|f\| = 1$,

$$\mathbb{P}\Big(\log |\langle f, G_n v \rangle| \in [a, a+\Delta] + nq \Big)$$
$$\sim \left(1 - e^{-s\Delta}\right) \frac{r_s(x) r_s^*(y)}{\nu_s(r_s)} \frac{\exp\left(-n\Lambda^*(q + \frac{a}{n})\right)}{s\sigma_s\sqrt{2\pi n}}.$$

- When $|a| = o(\sqrt{n})$, the exponential term can be written as
$$\exp\left(-n\Lambda^*\left(q + \frac{a}{n}\right)\right) \sim e^{-sa} e^{-n\Lambda^*(q)}.$$
- We also obtained similar results for $s \in (-s_0, 0)$.

Outline

Key ideas of the proof: (1)

Recall: the SLLN, CLT, LIL, LD bounds are proved using a comparison between $|\langle f, G_n v\rangle|$ and $\|G_n v\|$: e.g., in the proof of SLLN, one uses $\frac{1}{n^{\varepsilon}} \log \frac{|\langle f, G_n v\rangle|}{\|G_n v\|} \to 0$, a.s.

Classical approach to establish precise LD: prove an Edgeworth expansion under the changed measure, cf. Bahadur-Rao (1960), Petrov (1965), Dembo-Zeitouni (1998).

Our approach:

Step 1 (Decomposition):

- We start with an exact decomposition: for any $x = \mathbb{R}v \in \mathbb{P}(V)$ and $y = \mathbb{R}f \in \mathbb{P}(V^*)$ with $\|v\| = \|f\| = 1$,

$$\begin{aligned} \log |\langle f, G_n v\rangle| &= \log \|G_n v\| + \log \delta(y, G_n \cdot x) \\ &= \sigma(G_n, x) + \log \delta(y, X_n^x), \end{aligned}$$

  where $\delta(y, x) = \frac{|\langle f, v\rangle|}{\|f\|\|v\|}$.

- We would like to use LD asymptotics or the Edgeworth expansion for the couple $(X_n^x, \log \|G_n v\|)$ with target functions (cf. Grama-Liu-X., SPA 2021, JEMS 2022).

  The difficulty: the Markov chain X_n^x may stay in or very close to the hyperplane $\ker f = \{x \in \mathbb{P}(V) : \delta(y, x) = 0\}$, where $y = \mathbb{R}f \in \mathbb{P}(V^*)$.

Key ideas of the proof: (2a)

Step 2 (Discretization–we discretize the function $\log|\delta(y,\cdot)|$):

Let $U(t) = t$ for $t \in [0,1]$, $U(t) = 0$ for $t < 0$ and $U(t) = 1$ for $t > 1$. Let $\eta \in (0, \frac{1}{2}]$ be a constant. For any integer $k \geq 0$, define

$$U_k(t) = U\left(\frac{t - \eta(k-1)}{\eta}\right), \quad h_k(t) = U_k(t) - U_{k+1}(t), \quad t \in \mathbb{R}.$$

For any $x \in \mathbb{P}(V)$ and $y \in \mathbb{P}(V^*)$, set

$$\chi_k^y(x) = h_k(-\log\delta(y,x)) \quad \text{and} \quad \overline{\chi}_k^y(x) = U_k(-\log\delta(y,x)).$$

We have the following partition of the unity on $\mathbb{P}(V)$:

$$\sum_{k=0}^{\infty} \chi_k^y(x) = 1, \quad \sum_{k=0}^{M_n-1} \chi_k^y(x) + \overline{\chi}_{M_n}^y(x) = 1. \tag{3.1}$$

Key ideas of the proof: (2b)

Lemma 1 (Grama-Liu-X., AOP 2023)

There exists a constant $c > 0$ such that for any $\gamma \in (0,1]$, $k \geq 0$ and $y \in \mathbb{P}(V^*)$, we have $\chi_k^y \in \mathscr{B}_\gamma$ and $\|\chi_k^y\|_\gamma \leq \frac{c e^{\gamma \eta k}}{\eta^\gamma}$.

Let $M_n = \lfloor A \log n \rfloor$ with $A > 0$. Denote

$$\varphi_{s,k}^y = r_s^{-1} \chi_k^y \quad \text{for} \quad 0 \leq k \leq M_n - 1, \qquad \varphi_{s,M_n}^y = r_s^{-1} \overline{\chi}_{M_n}^y. \tag{3.2}$$

Then $r_s^{-1} = \sum_{k=0}^{M_n} \varphi_{s,k}^y$.

Key ideas of the proof: (3a)

Step 3 (A change of measure and smoothing techniques):

Denote $T_n^v = \log \|G_n v\| - nq$. By the change of measure formula,

$$\begin{aligned}
&\mathbb{P}\big(\log |\langle f, G_n v\rangle| \geq nq\big) \\
&= r_s(x) e^{-n\Lambda^*(q)} \mathbb{E}_{\mathbb{Q}_s^x}\Big[r_s^{-1}(X_n^x) e^{-sT_n^v} \mathbb{1}_{\{T_n^v + \log \delta(y, X_n^x) \geq 0\}}\Big] \\
&= r_s(x) e^{-n\Lambda^*(q)} \sum_{k=0}^{M_n} \mathbb{E}_{\mathbb{Q}_s^x}\Big[\varphi_{s,k}^y(X_n^x) e^{-sT_n^v} \mathbb{1}_{\{T_n^v + \log \delta(y, X_n^x) \geq 0\}}\Big] \\
&=: r_s(x) e^{-n\Lambda^*(q)} \sum_{k=0}^{M_n} F_{n,k}.
\end{aligned}$$

For $k = M_n$, we have $F_{n,M_n} = o(\frac{1}{\sqrt{n}})$.
For $0 \leq k \leq M_n - 1$, since $\log \delta(y,x) \leq -\eta(k+1)$ when $x \in \operatorname{supp} \varphi_{s,k}^y$, we get

$$\sum_{k=0}^{M_n-1} F_{n,k} \leq \sum_{k=0}^{M_n-1} \mathbb{E}_{\mathbb{Q}_s^x}\Big[\varphi_{s,k}^y(X_n^x) e^{-sT_n^v} \mathbb{1}_{\{T_n^v - \eta(k+1) \geq 0\}}\Big] =: \sum_{k=0}^{M_n-1} H_{n,k}. \tag{3.3}$$

Key ideas of the proof: (3b)

We then use smoothing techniques and the Fourier inversion formula:

Fix a non-negative density function ρ on $\mathbb{R}$ with $\int_{\mathbb{R}} \rho(u)du = 1$, whose Fourier transform $\widehat{\rho}$ is supported on $[-1, 1]$. Moreover, we take ρ such that there exists a constant $c > 0$ such that $\rho(u) \leq \frac{c}{1+|u|^p}$ for all $p > 1$ and $u \in \mathbb{R}$.

For any $\varepsilon > 0$, define the scaled density function ρ_ε by $\rho_\varepsilon(u) = \frac{1}{\varepsilon}\rho(\frac{u}{\varepsilon})$, $u \in \mathbb{R}$.

Using the Fourier inversion formula, we obtain

$$\sum_{k=0}^{M_n-1} H_{n,k} \approx \frac{1}{2\pi} \sum_{k=0}^{M_n-1} e^{-s\eta(k+1)} \int_{\mathbb{R}} e^{-it\eta(k+1)} R^n_{s,it}\big(\varphi^y_{s,k}\big)(x)\widehat{\Psi}^+_{s,\eta,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt$$

where the perturbed transfer operator $R_{s,it}$ pops out.

Key ideas of the proof: (4a)

Step 4 (Using spectral gap properties of $R_{s,it}$):

1. The perturbed operator $R_{s,it}$ is the Fourier transform of the norm cocycle under the changed measure $\mathbb{Q}_s^x$: with $q = \Lambda'(s)$,

$$R_{s,it}\varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x}\left[e^{it(\sigma(g,x)-q)}\varphi(g\cdot x)\right].$$

2. Spectral gap: for small $\delta > 0$ and $t \in (-\delta, \delta)$,

$$R^n_{s,it} = \lambda^n_{s,it}\Pi_{s,it} + N^n_{s,it},$$

where

$$\lambda_{s,it} := e^{-itq}\frac{\kappa(s+it)}{\kappa(s)} = 1 - \frac{\sigma_s^2}{2}t^2 - i\frac{\Lambda'''(s)}{6}t^3 + o(t^3),$$

with $\sigma_s > 0$ (under condition (IP)).

3. $\Pi_{s,0}\varphi = \pi_s(\varphi)$. For any compact set $K \subset I_\mu^\circ$ and integer $k \geq 0$, there exist $C > 0$ and $0 < a < 1$ such that

$$\sup_{s\in K}\sup_{|t|<\delta}\|\frac{d^k}{dt^k}\Pi_{s,it}\|_{\mathscr{B}_\gamma\to\mathscr{B}_\gamma} \leq C,$$

$$\sup_{s\in K}\sup_{|t|<\delta}\|\frac{d^k}{dt^k}N_{s,it}^n\|_{\mathscr{B}_\gamma\to\mathscr{B}_\gamma} \leq Ca^n.$$

4. For any compact sets $K \subset I_\mu^\circ$ and $T \subseteq \mathbb{R}\backslash\{0\}$, there exists $C > 0$ such that

$$\sup_{s\in K}\sup_{t\in T}\sup_{x\in\mathbb{P}(V)}|R_{s,it}^n\varphi(x)| \leq e^{-Cn}\|\varphi\|_\gamma.$$

Proof strategy:

- Perturbation theorem applied to P_s, together with the relation between the operators $R_{s,it}$ and P_z;
- Lemma of Hennion and Hervé (2001): $\limsup_{t\to s}\varrho(P(t)) \leq \varrho(P(s))$.

Key ideas of the proof: (4b)

We then make use of the asymptotic expansion of the integral of $R_{s,it}$:

Proposition (Grama-Liu-X., JEMS 2022)

For any compact set $K \subset I_\mu^\circ$, uniformly in $s \in K$, $x \in \mathbb{P}(V)$, $|l| \leq \frac{1}{\sqrt{n}}$ and $\varphi \in \mathscr{B}_\gamma$,

$$\left| \sigma_s \sqrt{\frac{n}{2\pi}}\, e^{\frac{nl^2}{2\sigma_s^2}} \int_{\mathbb{R}} e^{-itln} R_{s,it}^n(\varphi)(x)\psi(t)dt - \psi(0)\pi_s(\varphi) \right|$$
$$\leq \frac{C}{\sqrt{n}}\|\varphi\|_\gamma + \frac{C}{n}\|\varphi\|_\gamma \sup_{|t|\leq\delta} \big(|\psi(t)| + |\psi'(t)|\big) + Ce^{-cn}\|\varphi\|_\gamma \int_{\mathbb{R}} |\psi(t)|dt.$$

Note that $\|\varphi_{s,k}^y\|_\gamma \to \infty$ as $k \to \infty$.

Key ideas of the proof: (5a)

Step 5 (Patch up the pieces using the Hölder regularity of the measure π_s):

Under the measure $\mathbb{Q}_s^x$, the Markov chain $(X_n^x)_{n\geq 0}$ has a unique invariant probability measure π_s on $\mathbb{P}(V)$.

Lemma (Grama-Liu-X., AOP 2023)

Let $K \subseteq I_\mu^\circ$ be any compact set. Then there exists a constant $c > 0$ such that for any $s \in K$ and $y \in \mathbb{P}(V^*)$,

$$\sum_{k=0}^{M_n-1} e^{-s\eta(k+1)} \pi_s\big(\varphi_{s,k}^y\big) \leq \int_{\mathbb{P}(V)} \delta(y,x)^s \, r_s^{-1}(x) \, \pi_s(dx)$$

and

$$\sum_{k=1}^{M_n-1} e^{-s\eta(k-1)} \pi_s\big(\varphi_{s,k}^y\big) \geq \int_{\mathbb{P}(V)} \delta(y,x)^s \, r_s^{-1}(x) \, \pi_s(dx) - \frac{c}{n^2}.$$

The proof of this lemma is based on the Hölder regularity of the invariant measure π_s, see the next page.

Key ideas of the proof: (5b)

Denote $B(y,r) = \{x \in \mathbb{P}(V) : \delta(y,x) \leq r\}$ for $y \in \mathbb{P}(V^*)$ and $r \geq 0$.

We can establish the following Hölder regularity of the invariant measure π_s:

Theorem (Grama-Liu-X., AOP 2023)

For any $s \in I_\mu^\circ$, there exists $\alpha > 0$ such that

$$\sup_{y \in \mathbb{P}(V^*)} \int_{\mathbb{P}(V)} \frac{1}{\delta(y,x)^\alpha} \pi_s(dx) < +\infty.$$

In particular, there exist $\alpha, c > 0$ such that for any $0 < r < 1$,

$$\sup_{y \in \mathbb{P}(V^*)} \pi_s\big(B(y,r)\big) \leq c r^\alpha.$$

Bourgain J.: Finitely supported measures on $SL_2(\mathbb{R})$ which are absolutely continuous at infinity. Geometric aspects of functional analysis, 133-141, 2012.

Outline

Potential applications of LD for the coefficients

1. Multi-type branching processes in random environments.
2. Limit theorems for first passage times of multivariate perpetuity sequences initiated by Kesten (Acta Math. 1973).

   Perpetuity sequences arise in the ARCH and GARCH financial time series models, branching processes and branching random walks, and can be applied to the Quicksort algorithm in computer science and recently to modern machine learning.

Multivariate perpetuity sequences

1. We are interested in the tail behaviors of the multivariate perpetuity sequence: with $V_1 = Q_1$ and $n \geq 2$,

$$V_n = Q_1 + M_1 Q_2 + \cdots + (M_1 \dots M_{n-1}) Q_n,$$

where $(M_n, Q_n)_{n\geq 1}$ is an i.i.d. sequence, M_1 is a $d \times d$ random matrix with nonnegative entries, and Q_1 is a nonnegative random vector in $\mathbb{R}^d$.

2. If $\mathbb{E}(\log \|M_1\|) + \mathbb{E}(\log \|Q_1\|) < \infty$ and the first Lyapunov exponent λ of $(M_n)_{n\geq 1}$ is negative, then V_n converges almost surely to the random variable

$$V = Q_1 + \sum_{n=2}^{\infty} (M_1 \dots M_{n-1}) Q_n,$$

which satisfies the random difference equation

$$V \stackrel{d}{=} MV + Q,$$

where (M, Q) is an independent copy of (M_1, Q_1), and V is independent of (M, Q).

Multivariate perpetuity sequences

1. We study the asymptotic properties of the first passage time

$$\tau_u^y = \inf\{n \geq 1 : \langle y, V_n \rangle > u\}, \quad u \to +\infty,$$

   where $y \in \mathbb{S}_+^{d-1} = \{v \in \mathbb{R}_+^d : \|v\| = 1\}$.
2. Kesten (1973, Acta Math.) exhibited the heavy-tailed characteristics of V: under suitable conditions on (M_1, Q_1), there exist constants $\mathscr{C}_y > 0$ and $\alpha > 0$ such that

$$\mathbb{P}\left(\tau_u^y < \infty\right) = \mathbb{P}(\langle y, V \rangle > u) \sim \mathscr{C}_y u^{-\alpha}, \quad \text{as } u \to +\infty.$$

Weak law of large numbers

For any $s \in I_\mu$, define $\kappa(s) = \lim_{n\to\infty} \left(\mathbb{E}\|G_n\|^s\right)^{\frac{1}{n}}$ and $\Lambda = \log \kappa$. Assume that there exists a constant $\alpha \in I_\mu^\circ$ such that

$$\Lambda(\alpha) = 0. \tag{4.1}$$

Set $\rho = \Lambda'(\alpha)$.

Theorem (Weak law of large numbers, Mentemeier-X., arXiv 2023)

For any $\varepsilon > 0$, we have

$$\lim_{u\to\infty} \mathbb{P}\left(\left| \frac{\tau_u^y}{\log u} - \rho \right| > \varepsilon \,\middle|\, \tau_u^y < \infty \right) = 0.$$

Central limit theorem

Notation: $\rho = \Lambda'(\alpha) > 0$ and $\sigma_\alpha^2 = \Lambda''(\alpha) > 0$.

Theorem (Mentemeier-X., arXiv 2023)

For any $t \in \mathbb{R}$, we have

$$\lim_{u\to\infty} \mathbb{P}\left(\frac{\tau_u^y - \rho \log u}{\sigma_\alpha \rho^{3/2} \sqrt{\log u}} \leq t \,\middle|\, \tau_u^y < \infty \right) = \Phi(t).$$

The Petrov type large deviation asymptotics for the coefficients play a crucial role.

We have also established

1. precise large deviation asymptotics $\mathbb{P}(\tau_u^y \leq \beta \log u)$, where $\beta \in (0, \rho)$.
2. pointwise asymptotics $\mathbb{P}(\tau_u^y = [\beta \log u])$, where $\beta \in (0, \rho)$.

These results extend those of Buraczewski et al. (AOP 2016) from the one-dimensional setting to higher dimensions.

References

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Thank you!